Watershed cuts and Combinatorial Optimization

LPE-Coupure et Optimisation combinatoire

Laurent Najman[1],
Jean Cousty[1], Gilles Bertrand[1],Michel Couprie[1]
Camille Couprie[1], Leo Grady[2], Hugues Talbot[1]

[1]LIGM, UPE-MLV [2]Siemens Corporate Research

Master Course
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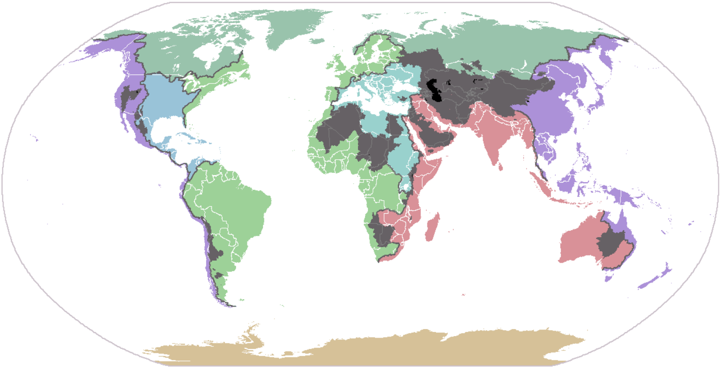
Outline

- Introduction
- Watershed cuts
  - Definition and consitency
  - Relative minimum spanning forests : watershed optimality
- Power Watersheds
  - A unifying framework for combinatorial optimization
  - The powerwatershed algorithm
  - Qualitative and quantitative comparison
- Conclusion and perspectives

Context



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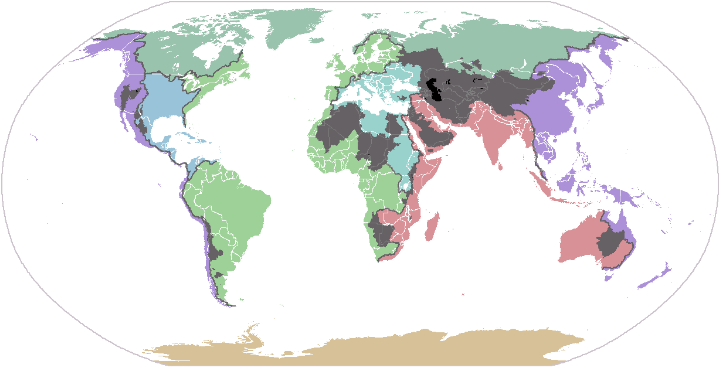

- For topographic purposes, the watershed has been studied since the 19th century (Maxwell, Jordan, . . .)

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- One hundred years later (1978), it was introduced by Digabel and Lantuéjoul for image segmentation

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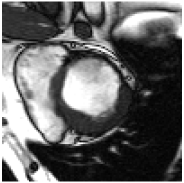

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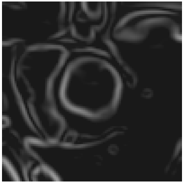

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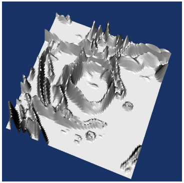

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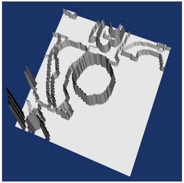

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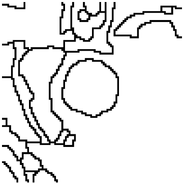
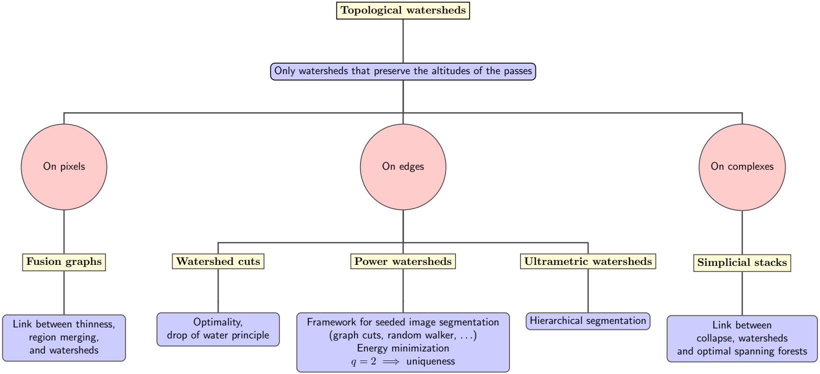


The family of discrete watersheds



In this talk

Problem

- *Watersheds in edge-weighted graphs ?*

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- *Mathematical properties ?*

In this talk

Problem

- *Watersheds in edge-weighted graphs ?*
- *Mathematical properties ?*
- *Use of watersheds for optimization ?*

Edge-weighted graph

- Let $G = (V, E)$ be a graph.
- Let F be a map from E to $\mathbb{R}$.

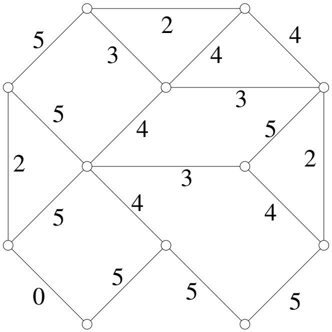
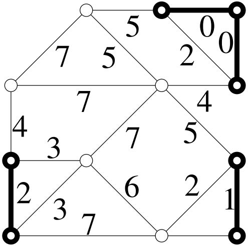


Image and edge-weighted graph

For applications to image analysis

- V is the set of *pixels*
- E corresponds to an *adjacency relation* on V, (*e.g.*, 4- or 8-adjacency in 2D)
- The altitude of u, an edge between two pixels x and y, represents the *dissimilarity between x and y*
  - $F(u) = |I(x) - I(y)|$.

Regional minima

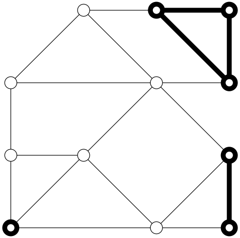


Definition

A subgraph X of G is a minimum of F (at altitude k) *if :*

- *X is connected ; and*
- *k is the altitude of any edge of X ; and*
- *the altitude of any edge adjacent to X is strictly greater than k*

Extension



a subgraph X

Definition (from Def. 12, (Ber05))

Let X and Y be two non-empty subgraphs of G.
We say that Y is an extension of X (in G) *if $X \subseteq Y$ and if any component of Y contains exactly one component of X.*

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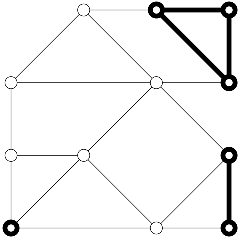

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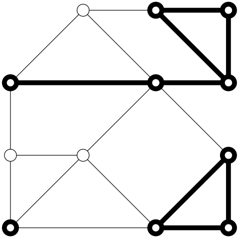

an extension Y of X

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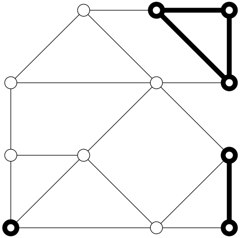

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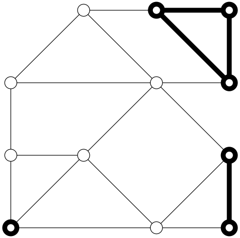

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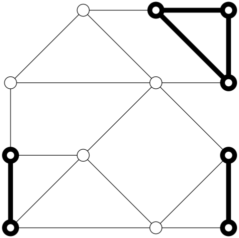

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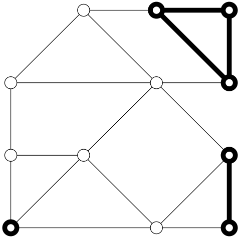

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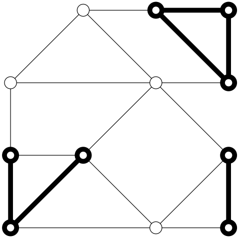

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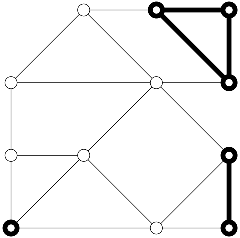

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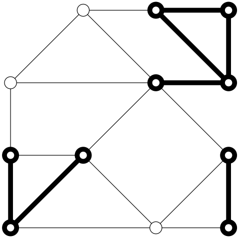

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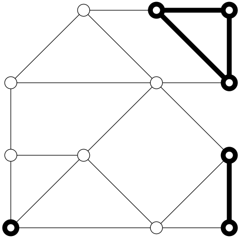

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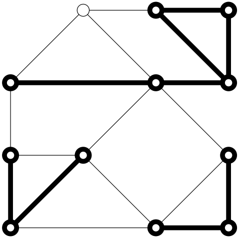

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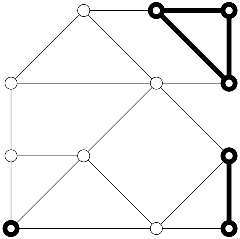

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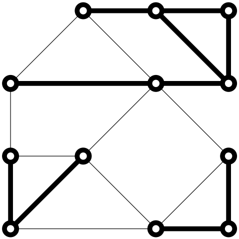

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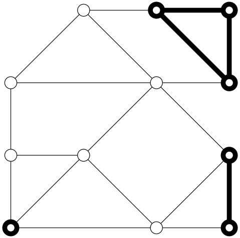

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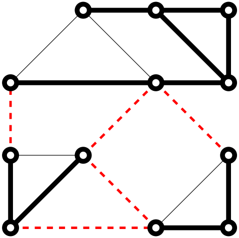

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Graph cut

Definition

Let X be a subgraph of G and let $S \subseteq E$ be an edge-set.

- *We say that S is a (graph) cut for X if $\overline{S}$ is an extension of X and if S is minimal for this property, i.e., if $T \subseteq S$ and $\overline{T}$ is an extension of X, then we have $T = S$.*

Watershed : intuitive idea

The church of Sorbier

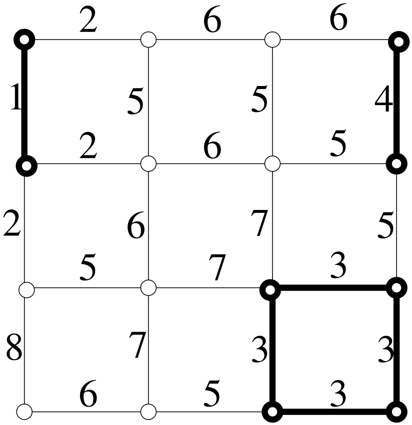
Watershed cut

Definition (drop of water principle)

Let $S \subseteq E$ be an edge-set.
We say that S is a watershed cut of F if $\overline{S}$ is an extension of $M(F)$ and if for any $u = \{x_0, y_0\} \in S$, there exist $\pi_1 = \langle x_0, \ldots, x_n \rangle$ and $\pi_2 = \langle y_0, \ldots, y_m \rangle$ which are two descending paths in $\overline{S}$ such that :

- *x_n and y_m are vertices of two distinct minima of F ; and*
- *$F(u) \geq F(\{x_0, x_1\})$ (resp. $F(u) \geq F(\{y_0, y_1\})$), whenever π_1 (resp. π_2) is not trivial.*

Watershed cut : example



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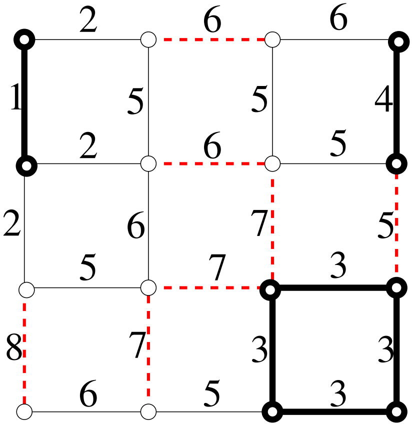

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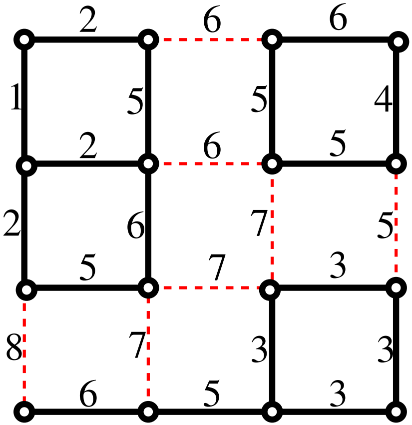

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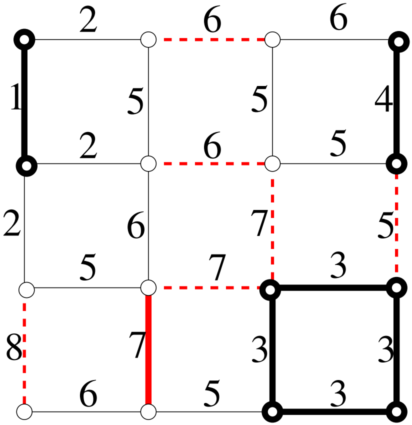

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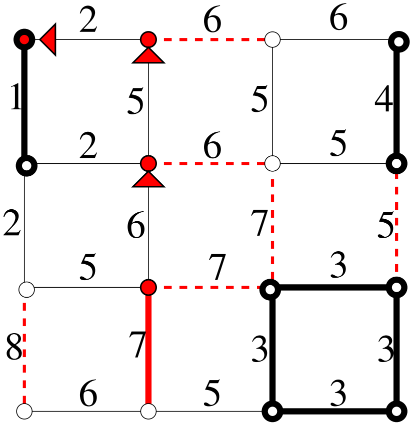

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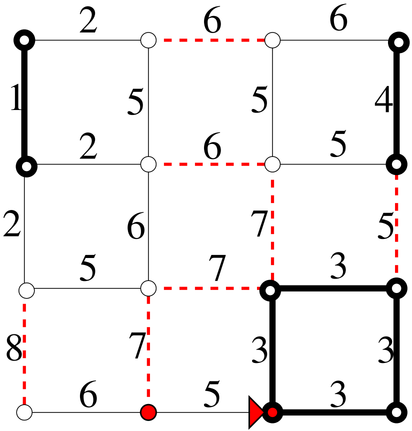

Catchment basins by a steepest descent property

- The *altitude* of a vertex x of G, denoted by $F(x)$, is the minimal altitude of an edge which contains x :
  - $F(x) = \min\{F(u) \mid u \in E, x \in u\}$

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  - $F(x) = \min\{F(u) \mid u \in E, x \in u\}$
- Let $\pi = \langle x_0, \ldots, x_l \rangle$ be a path in G. The path π is *a path with steepest descent for F* if, for any $i \in [1, l]$, $F(\{x_{i-1}, x_i\}) = F(x_{i-1})$.

Catchment basins by a steepest descent property

Definition

Let S be a cut for $M(F)$, the minima of F.
We say that S is a basin cut of F if, from each point of V to $M(F)$, there exists, in the graph induced by $\overline{S}$, a path with steepest descent for F.

Catchment basins by a steepest descent property

Theorem (consistency)

An edge-set $S \subseteq E$ is a basin cut of F if and only if *S is a watershed cut of F.*

Catchment basins by a steepest descent property

Theorem (consistency)

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Contribution

- As far as we know, in the literature about discrete watersheds, no similar property has ever been proved.

Relative forest

- Let X and Y be two non-empty subgraphs of G. We say that Y is a *forest relative to X* if :
  - Y is an extension of X ; and
  - for any extension $Z \subseteq Y$ of X, we have $Z = Y$ whenever $V(Z) = V(Y)$.

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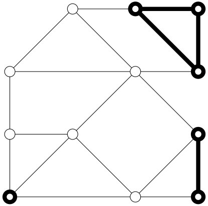

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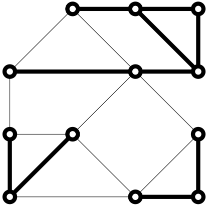

Minimum spanning forest

- The *weight of a forest* Y is the sum of its edge weights *i.e.*, $\sum_{u \in E(Y)} F(u)$.

Minimum spanning forest

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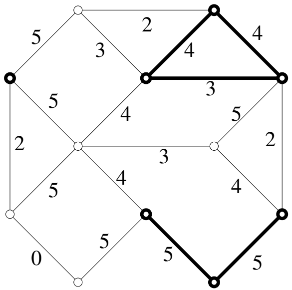
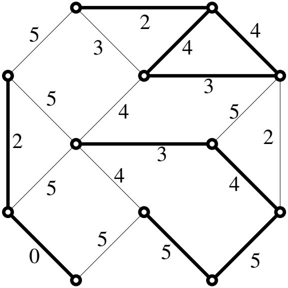
Definition

We say that Y is a minimum spanning forest (MSF) relative to X *if* Y *is a spanning forest relative to* X *and if the weight of* Y *is less than or equal to the weight of any other spanning forest relative to* X.

Minimum spanning forest : example



Minimum spanning forest : example



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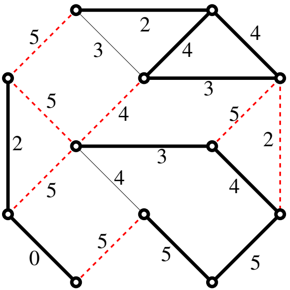

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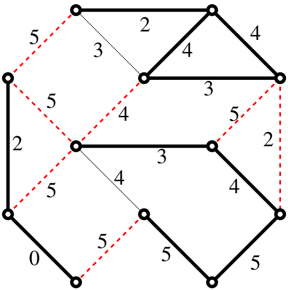

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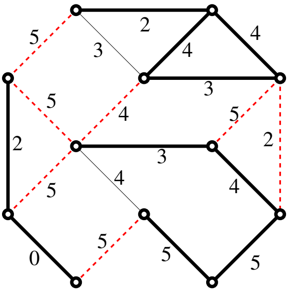
Minimum spanning forest : example



- If Y is a MSF relative to X, there exists a unique cut S for Y and this cut is also a cut for X ;
- In this case, we say that S is a *MSF cut for X*.

Watershed optimality

Theorem

An edge-set $S \subseteq E$ is a MSF cut for the minima of F if and only if *S is a watershed cut of F.*

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Contribution

- As far as we know, this is the first result which establishes watershed optimality.

Minimum spanning tree

- Computing a MSF $\Leftrightarrow$ computing a minimum spanning tree

Minimum spanning tree

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- Best algorithm [CHAZEL00] : quasi-linear time

Minimum spanning tree

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Problem

Can we reach a better complexity for computing watershed cuts ?

A linear-time algorithm for watershed cuts

Result

We propose the Stream Algorithm.

- *Stream Algorithm runs in linear time whatever the range of the input map*
  - *No need to sort*
  - *No need to use a hierarchical queue*

A linear-time algorithm for watershed cuts

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- *Furthermore, Stream Algorithm does not need to compute the minima as a pre-processing step.*

A linear-time algorithm for watershed cuts

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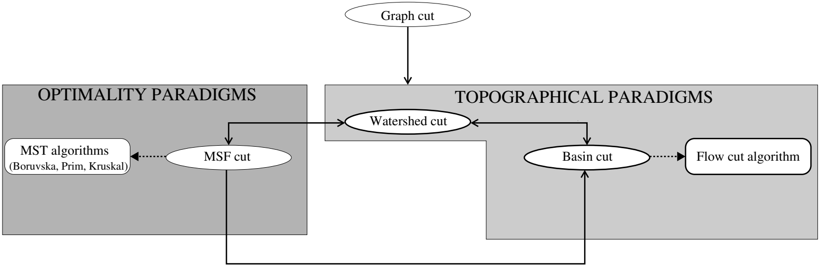
- *Stream Algorithm runs in linear time whatever the range of the input map*
  - *No need to sort*
  - *No need to use a hierarchical queue*
- *Furthermore, Stream Algorithm does not need to compute the minima as a pre-processing step.*

Contribution

To the best of our knowledge, this is the first watershed algorithm satisfying such properties

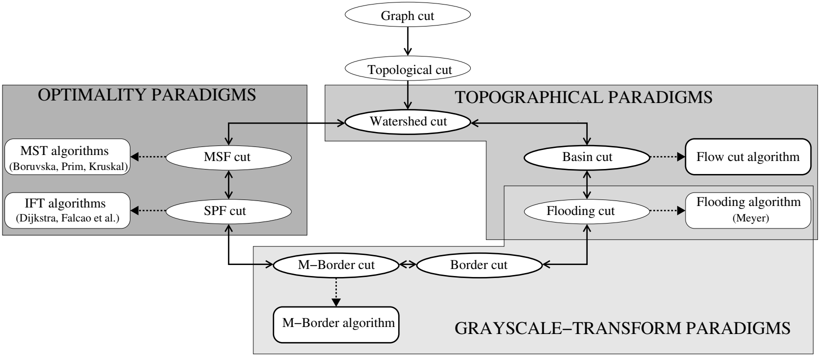
Conclusion on watershed cuts



Conclusion on watershed cuts

- In fact, there is more to say on watershed cuts . . .

Conclusion on watershed cuts



Watershed and optimization : intuitive idea

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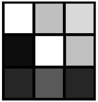
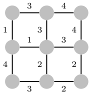
The church of Sorbier

Edge-weighted graph, revisited

- An image seen as a graph $G = (V, E)$

Image $3 \times 3 \rightarrow$ Weighted graph 3×3





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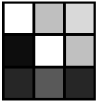
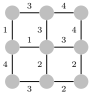


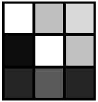


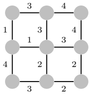
- Edges are weighted by a *similarity* measure
  i.e. *inversely* proportional to the image gradient
  - $w_{ij} = F(\{x_i, x_j\}) = F(u) = \exp(-\beta(I(x_i) - I(x_j))^2)$.

Edge-weighted graph, revisited

- An image seen as a graph $G = (V, E)$

Image $3 \times 3 \rightarrow$ Weighted graph 3×3





- Edges are weighted by a *similarity* measure
  i.e. *inversely* proportional to the image gradient
  - $w_{ij} = F(\{x_i, x_j\}) = F(u) = \exp(-\beta(I(x_i) - I(x_j))^2)$.
- Seed specification : 

New framework for image segmentation

- Given $\begin{cases} \text{two real positive numbers } p \text{ and } q \\ \text{seeds for the background } B, \\ \text{seeds for the foreground } F, \end{cases}$

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$$\min_x \sum_{e_{ij} \in E} w_{ij}^p |x_i - x_j|^q$$

Such that $x(F) = 1, x(B) = 0$.

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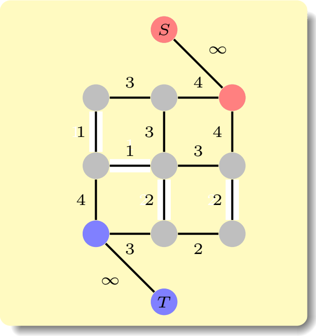
- Result : segmentation s defined $\forall i$ by $s_i = \begin{cases} 1 \text{ si } x_i \geq \frac{1}{2}, \\ 0 \text{ si } x_i < \frac{1}{2}. \end{cases}$

Graph Cuts

- Problem : compute x

$$x = \arg\min \sum_{e_{ij} \in E} {w_{ij}}^{p=1} |x_i - x_j|^{q=1}$$

- Min cut / Max flow duality
- Max Flow algorithm



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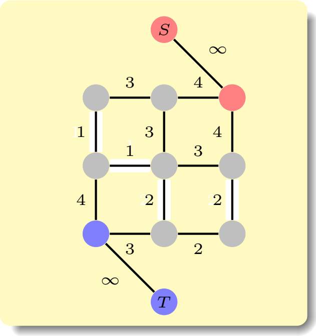

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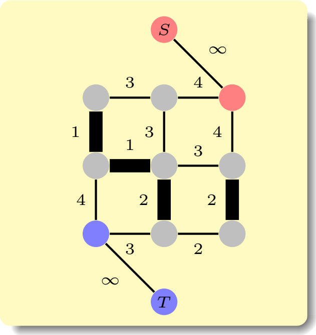

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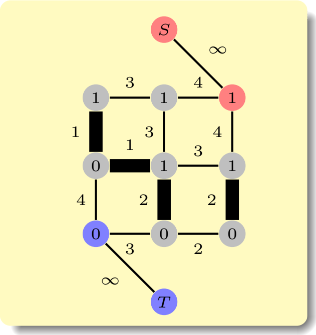
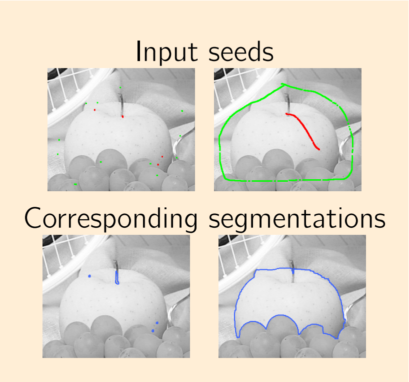


Graph Cuts : example

- favor small boundaries
- robust to uncentered seed placement



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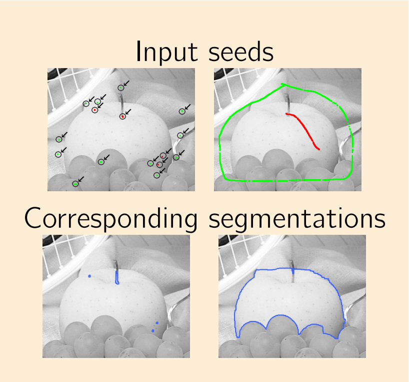
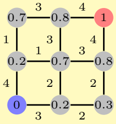


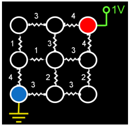
Random Walker

- Combinatorial version of the Dirichlet problem

$$x = \arg\min \sum_{e_{ij} \in E} w_{ij}(x_i - x_j)^2 \quad \leftarrow \quad u = \arg\min \int_\Omega |\nabla u|^2 d\Omega$$



- Potentials analogy



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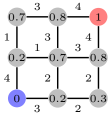

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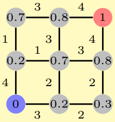

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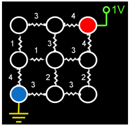

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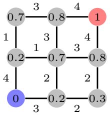

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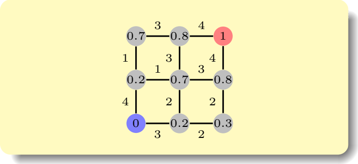

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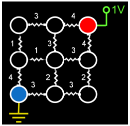

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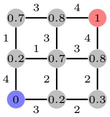

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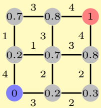
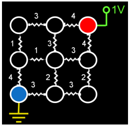


Stricly convex problem

$\Rightarrow$ unique optimal solution x^*

- Potentials analogy



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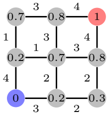
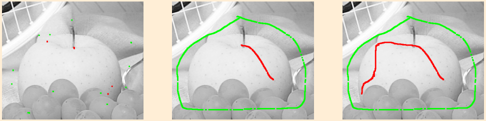


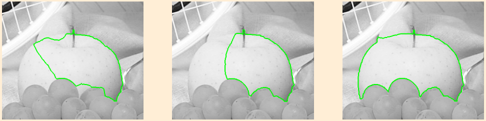
Random Walker : example

Input seeds



Corresponding probability/potential x

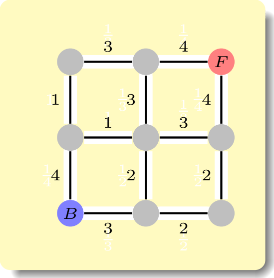
Corresponding segmentations (threshold of x)



Shortest path forest

- take the inverse of the weights
- the shortest path starting from each node to reach a seed node is computed
- Dijsktra algorithm
- [Sinop et al. 07] :

$$\lim_{p=q\to\infty} \bar{x}_{pq} = \min_x \sum_{e_{ij}\in E} {w_{ij}}^{p=q} (x_i - x_j)^q$$



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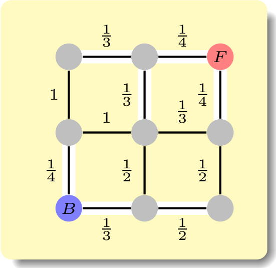

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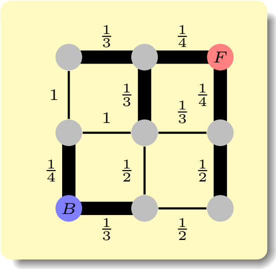
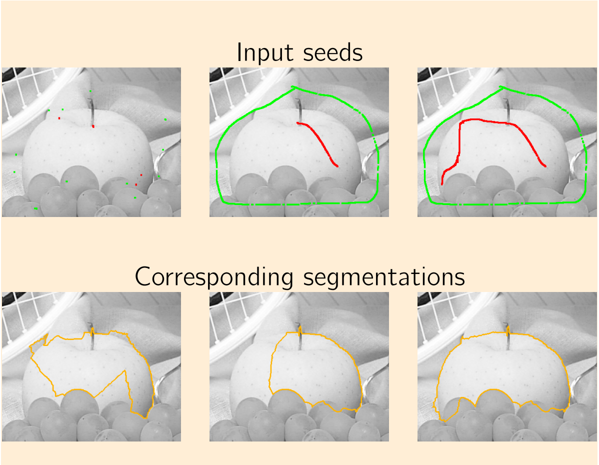


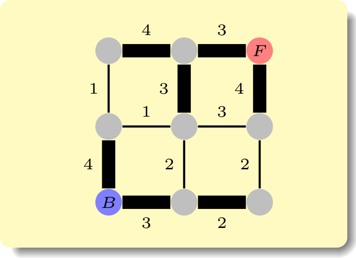
Shortest path forest : example

- Very sensitive to the object centering relatively to the seeds



Maximum Spanning Forest (MSF)

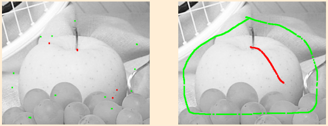
- maximize the sum of weights over the edges of a forest spanning the graph
- different labeled nodes have to belong to different trees
- Kruskal, Prim algorithms



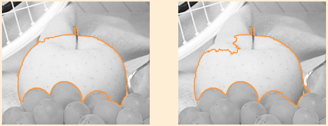
Maximum Spanning Forest (MSF) : example

- robust to small seeds
- leaking effect

Input seeds



Corresponding segmentations



Algorithms deriving from values of p et q

Recall the energy function : $\bar{x}_{pq} = \min_x \sum_{e_{ij} \in E} w_{ij}^p |x_i - x_j|^q$

q \ p	0	finite	∞
1	Reduction to seeds	Graph cuts	Max Spanning Forest [Allène et al. 07]
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Algorithm for the case $p = \infty$, variable q

$$\bar{x}_q = \lim_{p \to \infty} x^*_{p,q}$$

Power watershed algorithm (outline)

Build an MSF outside of plateaus, and optimize on plateaus

$$\sum_{e_{ij} \in \textbf{plateau}} |x_i - x_j|^q$$

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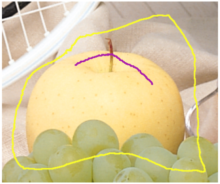
Theorem (Convergence)

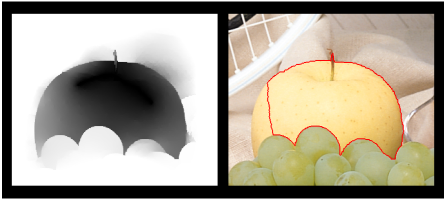
If $q > 1$, the potential $\bar{x}_{pq}$ converges, as $p \to \infty$, towards the potential $\bar{x}_q$ obtained by the Power Watershed algorithm.

Convergence of RW when $p \to \infty$ towards PW

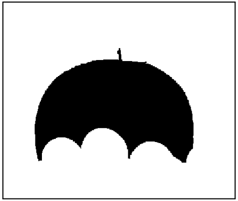
Input seeds



Random Walker $p = 1 \ldots 30$



PowerWatershed $q = 2$



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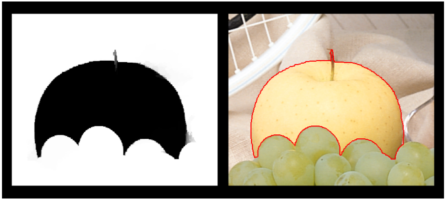


Theorems

Theorem (MSF cut)

The cut obtained by the Powerwatershed algorithm is a MSF cut.

Theorem (Watershed cut)

The cut obtained by the Powerwatershed algorithm is a watershed cut of the graph morphologically reconstructed from the seeds.

Theorem (Uniqueness)

When $q > 1$, the solution x^* to the minimization of

$$lim_{p \to \infty} \min_x \sum_{e_{ij} \in E} {w_{ij}}^p |x_i - x_j|^q$$

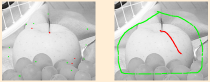
is unique.

(Thus, when $q > 1$, the solution $\bar{x}$ of the Powerwatershed algorithm is unique.)

Powerwatershed (q=2) : example

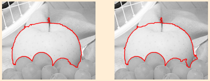
- robust to small seeds size
- less leaking than with standard Maximum Spanning Forest

Input seeds



Corresponding probability x

Corresponding segmentations (threshold of x)



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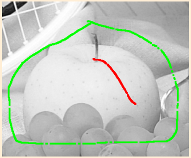
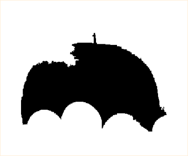


Powerwatershed ($q = 2$)

Prim algorithm for MSF



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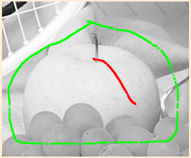

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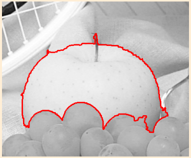

Prim algorithm for MSF

Generality of the framework

- Possibility to add unary terms to the energy function

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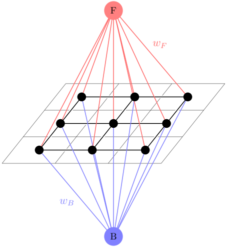

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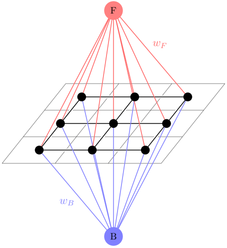
$$\min_x \sum_{e_{ij} \in E} w_{ij}^p |x_i - x_j|^q + \sum_{v_i} {w_{F_i}}^p |x_i - 1|^q + \sum_{v_i} {w_{Bi}}^p |x_i|^q$$



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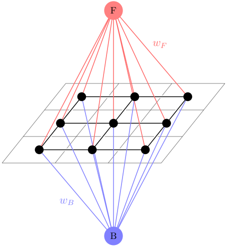
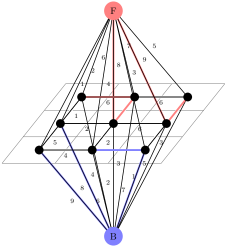
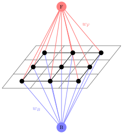




Generality of the framework

$$\min_x \sum_{e_{ij}\in E} w_{ij}^p |x_i - x_j|^q + \sum_{v_i} {w_{F_i}}^p |x_i - 1|^q + \sum_{v_i} {w_{B_i}}^p |x_i|^q$$



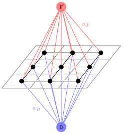
Image

Graph Cuts

MaxSF

Generality of the framework

$$\min_x \sum_{e_{ij}\in E} w_{ij}^p |x_i - x_j|^q + \sum_{v_i} {w_{F_i}}^p |x_i - 1|^q + \sum_{v_i} {w_{B_i}}^p |x_i|^q$$



Image

Graph Cuts

MaxSF

Contribution

To the best of our knowledge, this is the first time that watershed is used in other applications than seeded segmentation

Optimal multilabels segmentation

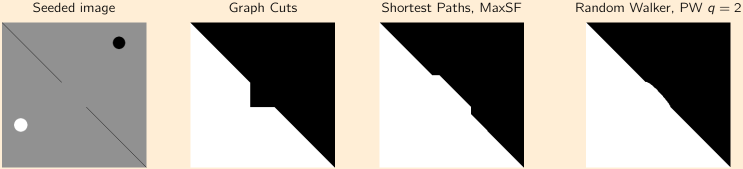
- More than 2-labels segmentation : NP-hard for Graph cuts
- Exact $n \geq 2$ labels segmentation for the other algorithms :
- n solutions $x^1, x^2, \ldots x^n$ computed
- x^k computed by enforcing $\begin{cases} x^k(n^k) = 1 \\ x^k(n^q) = 0 \text{ for all } q \neq k. \end{cases}$
- Each node i is affected to the label for which x_i^k is maximum :

$$s_i = \arg\max_k x_i^k$$

Input seeds

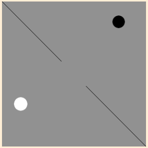
Segmentation by PowerWatershed ($q = 2$)

Algorithms behavior on plateaus

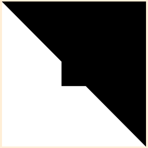


Algorithms behavior on plateaus

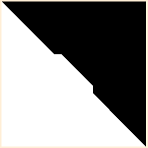
Seeded image



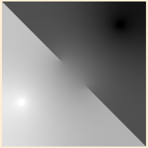
Graph Cuts



Shortest Paths, MaxSF

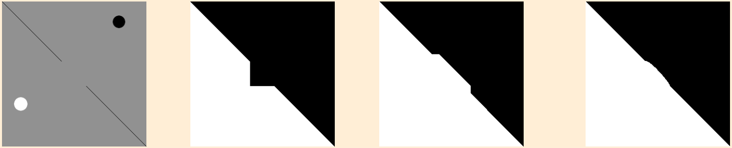


Random Walker, PW $q = 2$



Algorithms behavior on plateaus

Seeded image | Graph Cuts | Shortest Paths, MaxSF | Random Walker, PW $q = 2$

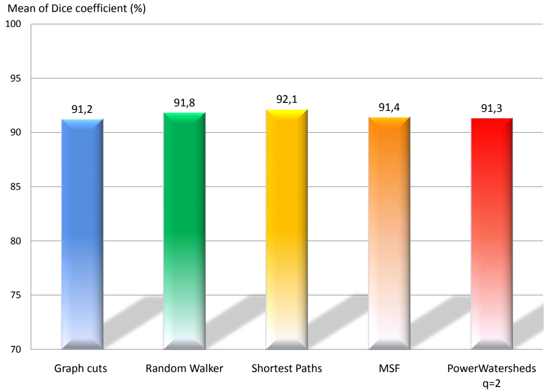


Algorithms comparison

- Evaluation on Berkeley database
- Ground truths
- 2 sets of seeds to study robustness to seeds centering :
  1. well centered seeds
  2. less centered seeds

Quantitative Results

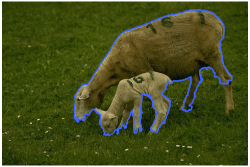
Dice coeff. between ground truths and the algorithms results on Berkeley database with the centered seeds.



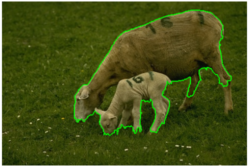
Examples

Input seeds

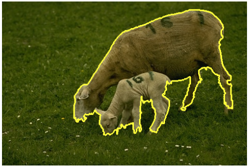
Graph Cuts



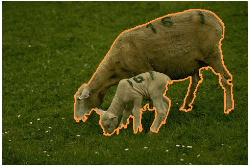
Random Walker



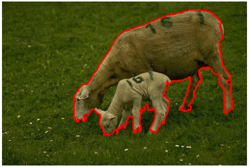
Shortest Paths



Max Spanning Forests

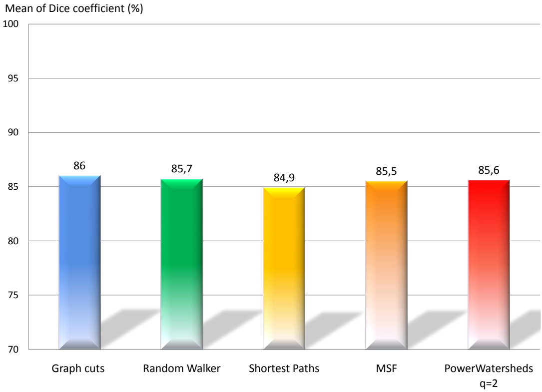


Power Watersheds $q = 2$



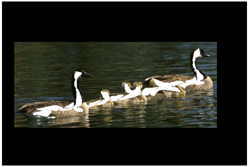
Quantitative Results

Dice coeff. between ground truths and the algorithms results on Berkeley database with the less centered seeds.

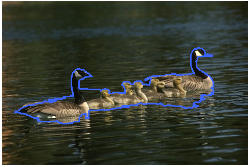


Examples

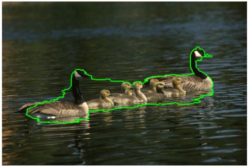
Input seeds



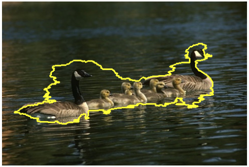
Graph Cuts



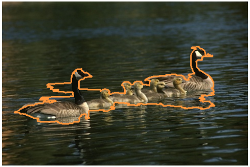
Random Walker



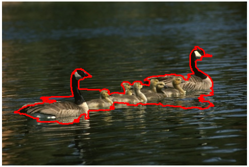
Shortest Paths



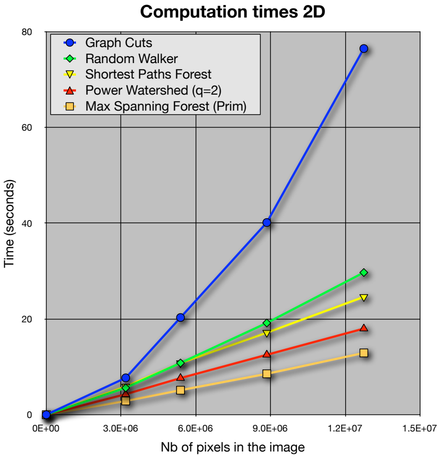
Max Spanning Forests



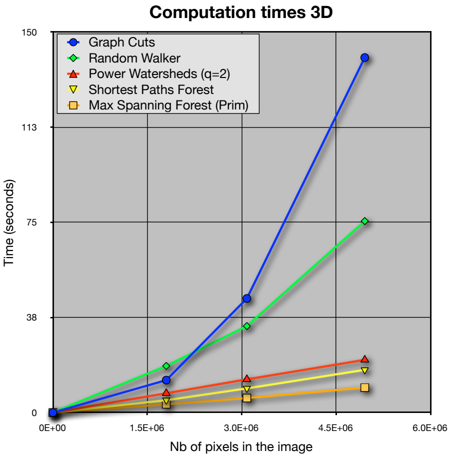
Power Watersheds $q = 2$



Computation time 2D



Computation time 3D



Which algorithm to use ?

- Graph Cuts :
  - good fit for 2D image segmentation in 2 labels
  - too slow for 3D segmentation
- Shortest Paths : segmentation of well centered seeds around the object
- Random Walker :
  - efficient with uncentered seeds
  - defined behavior on plateaus
- Max SF :
  - better segmentations than SPF with uncentered seeds
  - fast $\rightarrow$ 3D segmentation
- Powerwatershed $q = 2$:
  - MaxSF properties
  - less sensitive to leaking than standard MaxSF

Conclusion

- New framework unifying Graph Cuts, Random Walker, MSF and SPF.

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Contribution

The power watershed leads to a multilabel, scale and contrast invariant, unique global optimum obtained in practice in quasi-linear time

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The power watershed leads to a multilabel, scale and contrast invariant, unique global optimum obtained in practice in quasi-linear time

Non-convex diffusion using power watersheds

- Anisotropic diffusion [Perona-Malik 1990]

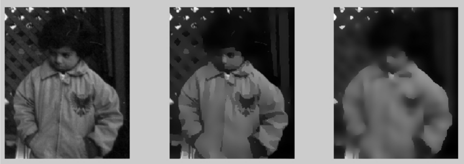


Image 100 iterations 200 iterations

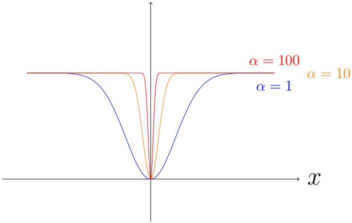
Goals of this work :

- perform anisotropic diffusion using an ℓ_0 norm to avoid the blurring effect
- optimize a non convex energy using Power Watershed [Couprie-Grady-Najman-Talbot, ICIP 2010]

Anisotropic diffusion and ℓ_0 norm

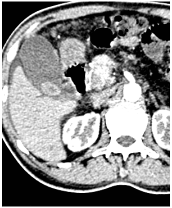
$$x^* = \arg\min_x \underbrace{\sum_{e_{ij}\in E} \sigma(x_i - x_j)}_{\text{smoothness term}} + \underbrace{\lambda \sum_{v_i \in V} \sigma(x_i - f_i)}_{\text{data fidelity term}}$$

$$\sigma(x) = 1 - \mathrm{e}^{-\alpha x^2}$$



Leads to piecewise constant results

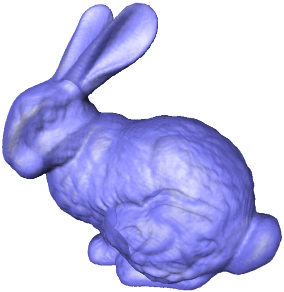
Original image



PW result

Surface reconstruction from a noisy set of dots

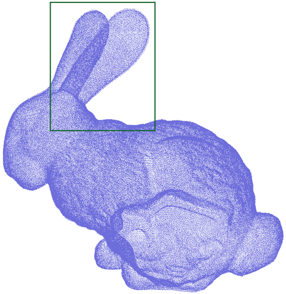
$\Rightarrow$



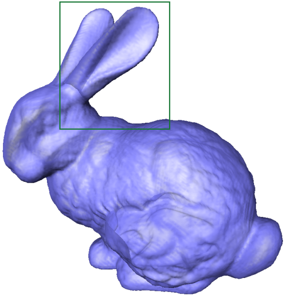
- Goal : given a noisy set of dots, find an explicit surface fitting the dots.

Joint work with Xavier Bresson

Surface reconstruction from a noisy set of dots



$\Rightarrow$



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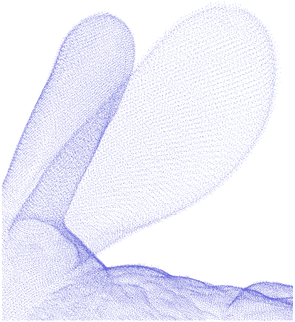
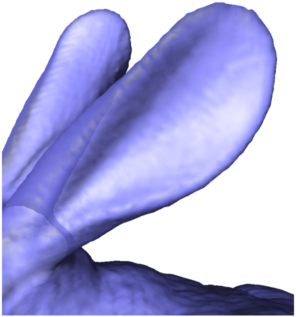
 ⇒ 

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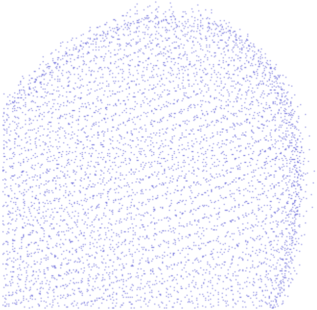

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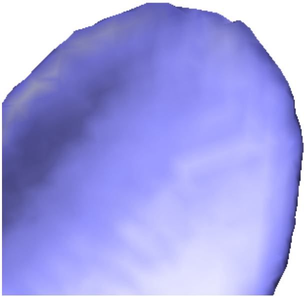


- Goal : given a noisy set of dots, find an explicit surface fitting the dots.

Joint work with Xavier Bresson

How to solve this problem

- Graph : 3D grid
- Here x represents the object indicator to recover.

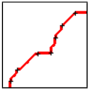
$$\bar{x} = \lim_{p \to \infty} \arg\min_x \sum_{e_{ij} \in E} {w_{ij}}^p |x_i - x_j|^q$$

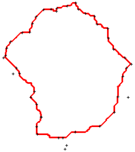
$$\text{s.t. } x(F) = 1, \; x(B) = 0$$

- weights : distance function from the set of dots to fit

Why PW are a good fit for this problem ?

numerous plateaus around the dots to fit $\rightarrow$ smooth isosurface is obtained





Power watershed solution

Perspectives

Future work

- Study of the different energies possibly minimized in this framework

Some papers on watershed cuts

Bibliography on watershed cuts

- Cousty, J., Bertrand, B., Najman, L. and Couprie, M. : Watershed cuts : minimum spanning forests and the drop of water principle. *IEEE Transactions on PAMI*, 31(8) :1362–1374, Aug. 2009.
- Cousty, J., Bertrand, G., Najman, L. and Couprie, M. : Watershed cuts : thinnings, shortest-path forests and topological watersheds. *IEEE Transactions on PAMI*, 32(5) :925-939, May 2010

Some papers on Power watersheds

Bibliography on powerwatersheds

- Couprie, C., Grady, L., Najman, L. and Talbot, H. :
  Power Watersheds : A Unifying Graph Based Optimization Framework.
  IEEE Transactions on PAMI, 33(7) :1384-1399, July 2011.
- C. Couprie, X. Bresson, L. Najman, H. Talbot and L. Grady : Surface reconstruction using Power watersheds. In *Proc. of ISMM 2011*.
- C. Couprie, L. Grady, L. Najman, and H. Talbot : Anisotropic diffusion using power watersheds. In *Proc. of ICIP 2010*.
- Couprie, C., Grady, L., Najman, L. and Talbot, H. :
  Power watersheds : A new image segmentation framework extending graph cuts, random walker and optimal spanning forest.
  In *Proc. of ICCV*, pages 731–738, Sept. 2009.

Questions

Source code available from

http ://sourceforge.net/projects/powerwatershed/